

Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under

$$S_3$$

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Abstract

In this paper, we derive eight basic identities of symmetry in three variables related to Bernoulli polynomials and power sums. These and most of their corollaries are new, since there have been results only about identities of symmetry in two variables. These abundance of symmetries shed new light even on the existing identities so as to yield some further interesting ones. The derivations of identities are based on the p -adic integral expression of the generating function for the Bernoulli polynomials and the quotient of integrals that can be expressed as the exponential generating function for the power sums.

Keywords: Bernoulli polynomial, power sum, Volkenborn integral, identities of symmetry.

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§I. Introduction and preliminaries

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p -adic integers, the field of p -adic rational numbers and

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the completion of the algebraic closure of \mathbb{Q}_p . For a uniformly differentiable (also called continuously differentiable) function $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ (cf. [4]), the Volkenborn integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z) d\mu(z) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j).$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1) d\mu(z) = \int_{\mathbb{Z}_p} f(z) d\mu(z) + f'(0). \quad (1)$$

Let $|\cdot|_p$ be the normalized absolute value of \mathbb{C}_p , such that $|p|_p = \frac{1}{p}$, and let

$$E = \{t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}}\}. \quad (2)$$

Then, for each fixed $t \in E$, the function $f(z) = e^{zt}$ is analytic on \mathbb{Z}_p and by applying (1) to this f , we get the p -adic integral expression of the generating function for Bernoulli numbers B_n :

$$\int_{\mathbb{Z}_p} e^{zt} d\mu(z) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (t \in E). \quad (3)$$

So we have the following p -adic integral expression of the generating function for the Bernoulli polynomials $B_n(x)$:

$$\int_{\mathbb{Z}_p} e^{(x+z)t} d\mu(z) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p). \quad (4)$$

Here and throughout this paper, we will have many instances to be able to interchange integral and infinite sum. That is justified by Proposition 55.4 in [4]. Let $S_k(n)$ denote the k -th power sum of the first $n+1$ nonnegative integers, namely

$$S_k(n) = \sum_{i=0}^n i^k = 0^k + 1^k + \cdots + n^k. \quad (5)$$

In particular,

$$S_0(n) = n+1, \quad S_k(0) = \begin{cases} 1, & \text{for } k=0, \\ 0, & \text{for } k>0. \end{cases} \quad (6)$$

From (3) and (5), one easily derives the following identities: for $w \in \mathbb{Z}_{>0}$,

$$\frac{w \int_{\mathbb{Z}_p} e^{xt} d\mu(x)}{\int_{\mathbb{Z}_p} e^{wyt} d\mu(y)} = \sum_{i=0}^{w-1} e^{it} = \sum_{k=0}^{\infty} S_k(w-1) \frac{t^k}{k!} \quad (t \in E). \quad (7)$$

In what follows, we will always assume that the Volkenborn integrals of the various exponential functions on \mathbb{Z}_p are defined for $t \in E$ (cf.(2)), and therefore it will not be mentioned.

[1]-[3], [5] and [6] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. For the brief history, one is referred to those papers.

In this paper, we will produce 8 basic identities of symmetry in three variables w_1, w_2, w_3 related to Bernoulli polynomials and power sums(cf. (44), (45), (48), (51), (55), (57), (59), (60)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature. These abundance of symmetries shed new light even on the existing identities. For instance, it has been known that (8) and (9) are equal(cf. [6, Cor.1], [3, Cor.2]) and (10) and (11) are so (cf. [3, (13)], [6, Cor.4]). In fact, (8)-(11) are all equal, as they can be derived from one and the same p -adic integral. Perhaps, this was neglected to mention in [3]. Also, we have a bunch of new identities in (12)-(15). All of these were obtained as corollaries (cf. Cor.9, 12, 15) to some of the basic identities by specializing the variable w_3 as 1. Those would not be unearthed if more symmetries had not been available.

$$\sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1} \quad (8)$$

$$= \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1} \quad (9)$$

$$= w_1^{n-1} \sum_{i=0}^{w_1-1} B_n(w_2 y_1 + \frac{w_2}{w_1} i) \quad (10)$$

$$= w_2^{n-1} \sum_{i=0}^{w_2-1} B_n(w_1 y_1 + \frac{w_1}{w_2} i) \quad (11)$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_1^{k+m-1} w_2^{k+\ell-1} \quad (12)$$

$$= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k(y_1 + \frac{i}{w_1}) S_{n-k}(w_2 - 1) w_2^{k-1} \quad (13)$$

$$= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k(y_1 + \frac{i}{w_2}) S_{n-k}(w_1 - 1) w_1^{k-1} \quad (14)$$

$$= (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} B_n(y_1 + \frac{i}{w_1} + \frac{j}{w_2}). \quad (15)$$

The derivations of identities is based on the p -adic integral expression of the generating function for the Bernoulli polynomials in (4) and the quotient of integrals in (7) that can be expressed as the exponential generating function for the power sums. We indebted this idea to the paper [3].

§II. Several types of quotients of Volkenborn integrals

Here we will introduce several types of quotients of Volkenborn integrals on \mathbb{Z}_p or \mathbb{Z}_p^3 from which some interesting identities follow owing to the built-in symmetries in w_1, w_2, w_3 . In the following, w_1, w_2, w_3 are positive integers and all of the explicit expressions of integrals in (17), (19), (21), and (23) are obtained from the identity in (3).

(a) Type Λ_{23}^i (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{23}^i) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} d\mu(x_1) d\mu(x_2) d\mu(x_3)}{(\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4))^i} \quad (16)$$

$$= \frac{(w_1 w_2 w_3)^{2-i} t^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (e^{w_1 w_2 w_3 t} - 1)^i}{(e^{w_2 w_3 t} - 1)(e^{w_1 w_3 t} - 1)(e^{w_1 w_2 t} - 1)}; \quad (17)$$

(b) Type Λ_{13}^i (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{13}^i) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} d\mu(x_1) d\mu(x_2) d\mu(x_3)}{(\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4))^i} \quad (18)$$

$$= \frac{(w_1 w_2 w_3)^{1-i} t^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (e^{w_1 w_2 w_3 t} - 1)^i}{(e^{w_1 t} - 1)(e^{w_2 t} - 1)(e^{w_3 t} - 1)}; \quad (19)$$

(c - 0) Type Λ_{12}^0

$$I(\Lambda_{12}^0) = \int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_2 w_3 y + w_1 w_3 y + w_1 w_2 y)t} d\mu(x_1) d\mu(x_2) d\mu(x_3) \quad (20)$$

$$= \frac{w_1 w_2 w_3 t^3 e^{(w_2 w_3 + w_1 w_3 + w_1 w_2)yt}}{(e^{w_1 t} - 1)(e^{w_2 t} - 1)(e^{w_3 t} - 1)}; \quad (21)$$

(c - 1) Type Λ_{12}^1

$$I(\Lambda_{12}^1) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3)t} d\mu(x_1) d\mu(x_2) d\mu(x_3)}{\int_{\mathbb{Z}_p^3} e^{(w_2 w_3 z_1 + w_1 w_3 z_2 + w_1 w_2 z_3)t} d\mu(z_1) d\mu(z_2) d\mu(z_3)} \quad (22)$$

$$= \frac{(w_1 w_2 w_3)^{-1} (e^{w_2 w_3 t} - 1) (e^{w_1 w_3 t} - 1) (e^{w_1 w_2 t} - 1)}{(e^{w_1 t} - 1)(e^{w_2 t} - 1)(e^{w_3 t} - 1)}. \quad (23)$$

All of the above p -adic integrals of various types are invariant under all permutations of w_1, w_2, w_3 , as one can see either from p -adic integral representations in (16), (18), (20), and (22) or from their explicit evaluations in (17), (19), (21), and (23).

§III. Identities for Bernoulli polynomials

(a - 0) First, let's consider Type Λ_{23}^i , for each $i = 0, 1, 2, 3$. The following results can be easily obtained from (4) and (7).

$$\begin{aligned} I(\Lambda_{23}^0) &= \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1)t} d\mu(x_1) \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2)t} d\mu(x_2) \int_{\mathbb{Z}_p} e^{w_1 w_2 (x_3 + w_3 y_3)t} d\mu(x_3) \\ &= \left(\sum_{k=0}^{\infty} \frac{B_k(w_1 y_1)}{k!} (w_2 w_3 t)^k \right) \left(\sum_{\ell=0}^{\infty} \frac{B_\ell(w_2 y_2)}{\ell!} (w_1 w_3 t)^\ell \right) \left(\sum_{m=0}^{\infty} \frac{B_m(w_3 y_3)}{m!} (w_1 w_2 t)^m \right) \end{aligned} \quad (24)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_2 y_2) B_m(w_3 y_3) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell} \right) \frac{t^n}{n!},$$

where the inner sum is over all nonnegative integers k, ℓ, m , with $k+\ell+m = n$, and

$$\binom{n}{k, \ell, m} = \frac{n!}{k! \ell! m!}. \quad (25)$$

(a - 1) Here we write $I(\Lambda_{23}^1)$ in two different ways:

$$\begin{aligned} (1) \quad I(\Lambda_{23}^1) &= \frac{1}{w_3} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu(x_1) \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2) t} d\mu(x_2) \\ &\quad \times \frac{w_3 \int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \quad (26) \\ &= \frac{1}{w_3} \left(\sum_{k=0}^{\infty} B_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} B_\ell(w_2 y_2) \frac{(w_1 w_3 t)^\ell}{\ell!} \right) \\ &\quad \times \left(\sum_{m=0}^{\infty} S_m(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_2 y_2) S_m(w_3 - 1) \right. \\ &\quad \left. \times w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell-1} \right) \frac{t^n}{n!}. \quad (27) \end{aligned}$$

(2) Invoking (7), (26) can also be written as

$$\begin{aligned} I(\Lambda_{23}^1) &= \frac{1}{w_3} \sum_{i=0}^{w_3-1} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu(x_1) \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2 + \frac{w_2}{w_3} i) t} d\mu(x_2) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{w_3} \sum_{i=0}^{w_3-1} \left(\sum_{k=0}^{\infty} B_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} B_\ell(w_2 y_2 + \frac{w_2}{w_3} i) \frac{(w_1 w_3 t)^\ell}{\ell!} \right) \\
&= \sum_{n=0}^{\infty} \left(w_3^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) \sum_{i=0}^{w_3-1} B_{n-k}(w_2 y_2 + \frac{w_2}{w_3} i) w_1^{n-k} w_2^k \right) \frac{t^n}{n!}. \quad (28)
\end{aligned}$$

(a - 2) Here we write $I(\Lambda_{23}^2)$ in three different ways:

$$\begin{aligned}
(1) \quad I(\Lambda_{23}^2) &= \frac{1}{w_2 w_3} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu(x_1) \times \frac{w_2 \int_{\mathbb{Z}_p} e^{w_1 w_3 x_2 t} d\mu(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \\
&\times \frac{w_3 \int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \quad (29)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{w_2 w_3} \left(\sum_{k=0}^{\infty} B_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} S_\ell(w_2 - 1) \frac{(w_1 w_3 t)^\ell}{\ell!} \right) \\
&\times \left(\sum_{m=0}^{\infty} S_m(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) S_\ell(w_2 - 1) S_m(w_3 - 1) \right. \\
&\left. \times w_1^{\ell+m} w_2^{k+m-1} w_3^{k+\ell-1} \right) \frac{t^n}{n!}. \quad (30)
\end{aligned}$$

(2) Invoking (7), (29) can also be written as

$$\begin{aligned}
I(\Lambda_{23}^2) &= \frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1 + \frac{w_1}{w_2} i) t} d\mu(x_1) \times \frac{w_3 \int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \quad (31) \\
&= \frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \left(\sum_{k=0}^{\infty} B_k(w_1 y_1 + \frac{w_1}{w_2} i) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} S_\ell(w_3 - 1) \frac{(w_1 w_2 t)^\ell}{\ell!} \right) \\
&= \sum_{n=0}^{\infty} \left(w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k(w_1 y_1 + \frac{w_1}{w_2} i) S_{n-k}(w_3 - 1) w_1^{n-k} w_3^{k-1} \right) \frac{t^n}{n!}. \quad (32)
\end{aligned}$$

(3) Invoking (7) once again, (31) can be written as

$$\begin{aligned}
I(\Lambda_{23}^2) &= \frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j)t} d\mu(x_1) \\
&= \frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \left(\sum_{n=0}^{\infty} B_n (w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j) \frac{(w_2 w_3 t)^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left((w_2 w_3)^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} B_n (w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j) \right) \frac{t^n}{n!}. \tag{33}
\end{aligned}$$

$(a - 3)$

$$\begin{aligned}
I(\Lambda_{23}^3) &= \frac{1}{w_1 w_2 w_3} \times \frac{w_1 \int_{\mathbb{Z}_p} e^{w_2 w_3 x_1 t} d\mu(x_1)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \times \frac{w_2 \int_{\mathbb{Z}_p} e^{w_1 w_3 x_2 t} d\mu(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \\
&\quad \times \frac{w_3 \int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \\
&= \frac{1}{w_1 w_2 w_3} \left(\sum_{k=0}^{\infty} S_k (w_1 - 1) \frac{(w_2 w_3 t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} S_{\ell} (w_2 - 1) \frac{(w_1 w_3 t)^{\ell}}{\ell!} \right) \\
&\quad \times \left(\sum_{m=0}^{\infty} S_m (w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k (w_1 - 1) S_{\ell} (w_2 - 1) S_m (w_3 - 1) \right. \\
&\quad \left. \times w_1^{\ell+m-1} w_2^{k+m-1} w_3^{k+\ell-1} \right) \frac{t^n}{n!}. \tag{34}
\end{aligned}$$

(b) For Type Λ_{13}^i ($i = 0, 1, 2, 3$), we may consider the analogous things to the ones in $(a - 0)$, $(a - 1)$, $(a - 2)$, and $(a - 3)$. However, these do not lead us to new identities. Indeed, if we substitute $w_2 w_3$, $w_1 w_3$, $w_1 w_2$ respectively for w_1, w_2, w_3 in (16), this amounts to replacing t by $w_1 w_2 w_3 t$ in (18). So, upon replacing w_1, w_2, w_3 respectively by $w_2 w_3, w_1 w_3, w_1 w_2$ and dividing by

$(w_1 w_2 w_3)^n$, in each of the expressions of (24), (27), (28), (30), (32)-(34), we will get the corresponding symmetric identities for Type Λ_{13}^i ($i = 0, 1, 2, 3$).

(c - 0)

$$\begin{aligned}
I(\Lambda_{12}^0) &= \int_{\mathbb{Z}_p} e^{w_1(x_1+w_2y)t} d\mu(x_1) \int_{\mathbb{Z}_p} e^{w_2(x_2+w_3y)t} d\mu(x_2) \int_{\mathbb{Z}_p} e^{w_3(x_3+w_1y)t} d\mu(x_3) \\
&= \left(\sum_{k=0}^{\infty} \frac{B_k(w_2y)}{k!} (w_1t)^k \right) \left(\sum_{\ell=0}^{\infty} \frac{B_\ell(w_3y)}{\ell!} (w_2t)^\ell \right) \left(\sum_{m=0}^{\infty} \frac{B_m(w_1y)}{m!} (w_3t)^m \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2y) B_\ell(w_3y) B_m(w_1y) w_1^k w_2^\ell w_3^m \right) \frac{t^n}{n!},
\end{aligned} \tag{35}$$

(c - 1)

$$\begin{aligned}
I(\Lambda_{12}^1) &= \frac{1}{w_1 w_2 w_3} \frac{w_2 \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu(x_1)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 z_3 t} d\mu(z_3)} \times \frac{w_3 \int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu(x_2)}{\int_{\mathbb{Z}_p} e^{w_2 w_3 z_1 t} d\mu(z_1)} \times \frac{w_1 \int_{\mathbb{Z}_p} e^{w_3 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} e^{w_3 w_1 z_2 t} d\mu(z_2)} \\
&= \frac{1}{w_1 w_2 w_3} \left(\sum_{k=0}^{\infty} S_k(w_2-1) \frac{(w_1t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} S_\ell(w_3-1) \frac{(w_2t)^\ell}{\ell!} \right) \\
&\quad \times \left(\sum_{m=0}^{\infty} S_m(w_1-1) \frac{(w_3t)^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_2-1) S_\ell(w_3-1) S_m(w_1-1) \right. \\
&\quad \left. \times w_1^{k-1} w_2^{\ell-1} w_3^{m-1} \right) \frac{t^n}{n!}.
\end{aligned} \tag{36}$$

§IV. Main theorems

As we noted earlier in the last paragraph of Section II, the various types of quotients of Volkenborn integrals are invariant under any permutation of w_1, w_2, w_3 . So the corresponding expressions in Section III are also invariant

under any permutation of w_1, w_2, w_3 . Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section III yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of S_3 . In particular, the number of possible distinct expressions are 1,2,3, or 6.(a-0),(a-1(1)),(a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 8 and 17, leaving the others as easy exercises for the reader. As for the case of Theorem 8, in addition to (50)-(52), we get the following three ones:

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) S_\ell(w_3 - 1) S_m(w_2 - 1) w_1^{\ell+m} w_3^{k+m-1} w_2^{k+\ell-1}, \quad (37)$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) S_\ell(w_1 - 1) S_m(w_3 - 1) w_2^{\ell+m} w_1^{k+m-1} w_3^{k+\ell-1}, \quad (38)$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) S_\ell(w_2 - 1) S_m(w_1 - 1) w_3^{\ell+m} w_2^{k+m-1} w_1^{k+\ell-1}. \quad (39)$$

But, by interchanging ℓ and m , we see that (37), (38), and (39) are respectively equal to (50), (51), and (52).

As to Theorem 17, in addition to (60) and (61), we have:

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_2 - 1) S_\ell(w_3 - 1) S_m(w_1 - 1) w_1^{k-1} w_2^{\ell-1} w_3^{m-1}, \quad (40)$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_3 - 1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_2^{k-1} w_3^{\ell-1} w_1^{m-1}, \quad (41)$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_3 - 1) S_\ell(w_2 - 1) S_m(w_1 - 1) w_1^{k-1} w_3^{\ell-1} w_2^{m-1}, \quad (42)$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_2 - 1) S_\ell(w_1 - 1) S_m(w_3 - 1) w_3^{k-1} w_2^{\ell-1} w_1^{m-1}. \quad (43)$$

However, (40) and (41) are equal to (60), as we can see by applying the permutations $k \rightarrow \ell, \ell \rightarrow m, m \rightarrow k$ for (40) and $k \rightarrow m, \ell \rightarrow k, m \rightarrow \ell$ for (41). Similarly, we see that (42) and (43) are equal to (61), by applying permutations $k \rightarrow \ell, \ell \rightarrow m, m \rightarrow k$ for (42) and $k \rightarrow m, \ell \rightarrow k, m \rightarrow \ell$ for (43).

THEOREM 1. *Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3 , so that it gives us six symmetries.*

$$\begin{aligned}
& \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_2 y_2) B_m(w_3 y_3) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_3 y_2) B_m(w_2 y_3) w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_1 y_2) B_m(w_3 y_3) w_2^{\ell+m} w_1^{k+m} w_3^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_3 y_2) B_m(w_1 y_3) w_2^{\ell+m} w_3^{k+m} w_1^{k+\ell} \tag{44} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_1 y_2) B_m(w_2 y_3) w_3^{\ell+m} w_1^{k+m} w_2^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_2 y_2) B_m(w_1 y_3) w_3^{\ell+m} w_2^{k+m} w_1^{k+\ell}.
\end{aligned}$$

THEOREM 2. *Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3 , so that it gives us six symmetries.*

$$\begin{aligned}
& \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_2 y_2) S_m(w_3 - 1) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_3 y_2) S_m(w_2 - 1) w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell-1} \tag{45} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_1 y_2) S_m(w_3 - 1) w_2^{\ell+m} w_1^{k+m} w_3^{k+\ell-1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_3 y_2) S_m(w_1 - 1) w_2^{\ell+m} w_3^{k+m} w_1^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_2 y_2) S_m(w_1 - 1) w_3^{\ell+m} w_2^{k+m} w_1^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_1 y_2) S_m(w_2 - 1) w_3^{\ell+m} w_1^{k+m} w_2^{k+\ell-1}.
\end{aligned}$$

Putting $w_3 = 1$ in (45), we get the following corollary.

COROLLARY 3. Let w_1, w_2 be any positive integers.

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) B_{n-k}(w_2 y_2) w_1^{n-k} w_2^k \\
&= \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) B_{n-k}(w_1 y_2) w_2^{n-k} w_1^k \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) B_\ell(w_2 y_2) S_m(w_1 - 1) w_2^{k+m} w_1^{k+\ell-1} \quad (46) \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(y_2) S_m(w_1 - 1) w_2^{\ell+m} w_1^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) B_\ell(w_1 y_2) S_m(w_2 - 1) w_1^{k+m} w_2^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(y_2) S_m(w_2 - 1) w_1^{\ell+m} w_2^{k+\ell-1}.
\end{aligned}$$

Letting further $w_2 = 1$ in (46), we have the following corollary.

COROLLARY 4. Let w_1 be any positive integer.

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) B_{n-k}(y_2) w_1^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} B_k(y_1) B_{n-k}(w_1 y_2) w_1^k \quad (47)
\end{aligned}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) B_\ell(y_2) S_m(w_1 - 1) w_1^{k+\ell-1}.$$

THEOREM 5. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3 , so that it gives us six symmetries.

$$\begin{aligned} & w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_3 y_1) \sum_{i=0}^{w_1-1} B_{n-k}(w_2 y_2 + \frac{w_2}{w_1} i) w_3^{n-k} w_2^k \\ &= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) \sum_{i=0}^{w_1-1} B_{n-k}(w_3 y_2 + \frac{w_3}{w_1} i) w_2^{n-k} w_3^k \\ &= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_3 y_1) \sum_{i=0}^{w_2-1} B_{n-k}(w_1 y_2 + \frac{w_1}{w_2} i) w_3^{n-k} w_1^k \quad (48) \\ &= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) \sum_{i=0}^{w_2-1} B_{n-k}(w_3 y_2 + \frac{w_3}{w_2} i) w_1^{n-k} w_3^k \\ &= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) \sum_{i=0}^{w_3-1} B_{n-k}(w_1 y_2 + \frac{w_1}{w_3} i) w_2^{n-k} w_1^k \\ &= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) \sum_{i=0}^{w_3-1} B_{n-k}(w_2 y_2 + \frac{w_2}{w_3} i) w_1^{n-k} w_2^k. \end{aligned}$$

Letting $w_3 = 1$ in (48), we obtain alternative expressions for the identities in (46).

COROLLARY 6. Let w_1, w_2 be any positive integers.

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) B_{n-k}(w_2 y_2) w_1^{n-k} w_2^k \\ &= \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) B_{n-k}(w_1 y_2) w_2^{n-k} w_1^k \quad (49) \\ &= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(y_1) \sum_{i=0}^{w_1-1} B_{n-k}(w_2 y_2 + \frac{w_2}{w_1} i) w_2^k \end{aligned}$$

$$\begin{aligned}
&= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) \sum_{i=0}^{w_1-1} B_{n-k}(y_2 + \frac{i}{w_1}) w_2^{n-k} \\
&= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(y_1) \sum_{i=0}^{w_2-1} B_{n-k}(w_1 y_2 + \frac{w_1}{w_2} i) w_1^k \\
&= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) \sum_{i=0}^{w_2-1} B_{n-k}(y_2 + \frac{i}{w_2}) w_1^{n-k}.
\end{aligned}$$

Putting further $w_2 = 1$ in (49), we have the alternative expressions for the identities for (47).

COROLLARY 7. *Let w_1 be any positive integer.*

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} B_k(y_1) B_{n-k}(w_1 y_2) w_1^k \\
&= \sum_{k=0}^n \binom{n}{k} B_k(y_2) B_{n-k}(w_1 y_1) w_1^k \\
&= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(y_1) \sum_{i=0}^{w_1-1} B_{n-k}(y_2 + \frac{i}{w_1}).
\end{aligned}$$

THEOREM 8. *Let w_1, w_2, w_3 be any positive integers. Then we have the following three symmetries in w_1, w_2, w_3 :*

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) S_\ell(w_2 - 1) S_m(w_3 - 1) w_1^{\ell+m} w_2^{k+m-1} w_3^{k+\ell-1} \quad (50)$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) S_\ell(w_3 - 1) S_m(w_1 - 1) w_2^{\ell+m} w_3^{k+m-1} w_1^{k+\ell-1} \quad (51)$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_3^{\ell+m} w_1^{k+m-1} w_2^{k+\ell-1}. \quad (52)$$

Putting $w_3 = 1$ in (50)-(52), we get the following corollary.

COROLLARY 9. Let w_1, w_2 be any positive integers.

$$\begin{aligned}
& \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1} \\
&= \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_1^{k+m-1} w_2^{k+\ell-1}.
\end{aligned} \tag{53}$$

Letting further $w_2 = 1$ in (53), we get the following corollary. This is also obtained in [6, Cor.2] and mentioned in [3].

COROLLARY 10. Let w_1 be any positive integer.

$$B_n(w_1 y_1) = \sum_{k=0}^n \binom{n}{k} B_k(y_1) S_{n-k}(w_1 - 1) w_1^{k-1}. \tag{54}$$

THEOREM 11. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3 , so that it gives us six symmetries.

$$\begin{aligned}
& w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k(w_2 y_1 + \frac{w_2}{w_1} i) S_{n-k}(w_3 - 1) w_2^{n-k} w_3^{k-1} \\
&= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k(w_3 y_1 + \frac{w_3}{w_1} i) S_{n-k}(w_2 - 1) w_3^{n-k} w_2^{k-1} \\
&= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k(w_1 y_1 + \frac{w_1}{w_2} i) S_{n-k}(w_3 - 1) w_1^{n-k} w_3^{k-1} \\
&= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k(w_3 y_1 + \frac{w_3}{w_2} i) S_{n-k}(w_1 - 1) w_3^{n-k} w_1^{k-1} \\
&= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} B_k(w_1 y_1 + \frac{w_1}{w_3} i) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1}
\end{aligned} \tag{55}$$

$$= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} B_k(w_2 y_1 + \frac{w_2}{w_3} i) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1}.$$

Putting $w_3 = 1$ in (55), we obtain the following corollary. In Section I, the identities in (53), (56), and (58) are combined to give those in (8)-(15).

COROLLARY 12. *Let w_1, w_2 be any positive integers.*

$$\begin{aligned} & w_1^{n-1} \sum_{i=0}^{w_1-1} B_n(w_2 y_1 + \frac{w_2}{w_1} i) \\ &= w_2^{n-1} \sum_{i=0}^{w_2-1} B_n(w_1 y_1 + \frac{w_1}{w_2} i) \\ &= \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_1) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1} \tag{56} \\ &= \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_1) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1} \\ &= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k(y_1 + \frac{i}{w_1}) S_{n-k}(w_2 - 1) w_2^{k-1} \\ &= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k(y_1 + \frac{i}{w_2}) S_{n-k}(w_1 - 1) w_1^{k-1}. \end{aligned}$$

Letting further $w_2 = 1$ in (56), we get the following corollary. This is the well-known multiplication formula for Bernoulli polynomials together with the relatively new identity mentioned in (54).

COROLLARY 13. *Let w_1 be any positive integer.*

$$\begin{aligned} B_n(w_1 y_1) &= w_1^{n-1} \sum_{i=0}^{w_1-1} B_n(y_1 + \frac{i}{w_1}) \\ &= \sum_{k=0}^n \binom{n}{k} B_k(y_1) S_{n-k}(w_1 - 1) w_1^{k-1}. \end{aligned}$$

THEOREM 14. Let w_1, w_2, w_3 be any positive integers. Then we have the following three symmetries in w_1, w_2, w_3 :

$$\begin{aligned}
& (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} B_n(w_3 y_1 + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j) \\
& = (w_2 w_3)^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} B_n(w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j) \\
& = (w_3 w_1)^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} B_n(w_2 y_1 + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j).
\end{aligned} \tag{57}$$

Letting $w_3 = 1$ in (57), we have the following corollary.

COROLLARY 15. Let w_1, w_2 be any positive integers.

$$\begin{aligned}
& w_1^{n-1} \sum_{j=0}^{w_1-1} B_n(w_2 y_1 + \frac{w_2}{w_1} j) \\
& = w_2^{n-1} \sum_{i=0}^{w_2-1} B_n(w_1 y_1 + \frac{w_1}{w_2} i) \\
& = (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} B_n(y_1 + \frac{i}{w_1} + \frac{j}{w_2}).
\end{aligned} \tag{58}$$

THEOREM 16. Let w_1, w_2, w_3 be any positive integers. Then we have the following two symmetries in w_1, w_2, w_3 :

$$\begin{aligned}
& \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y) B_\ell(w_2 y) B_m(w_3 y) w_3^k w_1^\ell w_2^m \\
& = \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y) B_\ell(w_3 y) B_m(w_2 y) w_2^k w_1^\ell w_3^m.
\end{aligned} \tag{59}$$

THEOREM 17. Let w_1, w_2, w_3 be any positive integers. Then we have the following two symmetries in w_1, w_2, w_3 :

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_1 - 1) S_\ell(w_2 - 1) S_m(w_3 - 1) w_3^{k-1} w_1^{\ell-1} w_2^{m-1} \tag{60}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} S_k(w_1 - 1) S_\ell(w_3 - 1) S_m(w_2 - 1) w_2^{k-1} w_1^{\ell-1} w_3^{m-1}. \quad (61)$$

Putting $w_3 = 1$ in (60) and (61) and multiplying the resulting identity by $w_1 w_2$, we get the following corollary.

COROLLARY 18. *Let w_1, w_2 be any positive integers.*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} S_k(w_2 - 1) S_{n-k}(w_1 - 1) w_1^k \\ &= \sum_{k=0}^n \binom{n}{k} S_k(w_1 - 1) S_{n-k}(w_2 - 1) w_2^k. \end{aligned}$$

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